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A note on yawed slender wings*

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SUMMARY

An integral equation of Abel type is derived for the wake strength in steady flow over a yawed slender wing. An analytic solution is obtained for a triangular plan form, and numerical results for other configurations.

1. Introduction

The original slender-wing theory of Jones [1] assumes an 'expanding' plan form, such that no wing-wake interaction occurs. This theory leads to a very simple conclusion, that the lift depends only on the properties of the section of greatest span. A similar result (derived here) applies for the roll moment, in laterally asymmetric flow.

In cases where wing-wake interaction occurs, e.g. when there is a highly-swept trailing edge, the slender-wing theory is less simple in its detail. For symmetric swallow-tailed swept wings, a number of authors (e.g. [2], [3]) have reduced the mathematical problem to a one-dimensional singular Volterra integral equation. The unknown quantity is, in effect, the strength of the wake vortex sheet, which must be determined section by section, starting at the point of commencement of the wake, and working downstream. The kernel of the integral equation depends on the nature of the cross-flow geometry; for doubly-connected geometries, such as the swallow-tailed airfoil, it involves elliptic functions.

Although there is some computational difficulty in solving such an integral equation, the problem is still far simpler than the full lifting-surface problem, at arbitrary aspect ratio. That problem involves a singular Fredholm integral equation in two dimensions, and is therefore far more demanding upon numerical accuracy. The mid-70's state of the art for lifting-surface computer programs is surveyed by Wang [4] and, although the achievements are impressive, there would still seem to be a place for the 'approximate but more accurate' (when there is adequate slenderness!) low-aspect-ratio limiting theories.

One important case, which does not appear to have been worked out in detail, is a yawed simple wing, as in Figure 1. In an (x, y, s) co-ordinate system, with y normal to the mean plane

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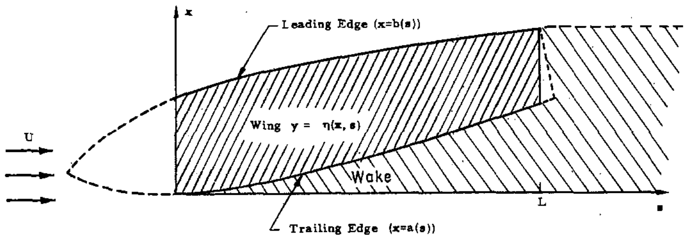

Figure 1. Sketch of yawed slender wing in plan.

of the wing, s in the streamwise direction, and x to starboard, we assume that the starboard edge

$$x = b(s), \qquad 0 < s < L, \tag{1.1}$$

is always a leading edge, whereas the port edge

$$x = a(s), \qquad 0 < s < L, \tag{1.2}$$

is always a trailing edge, and we set $a(o) = 0$. These conditions are satisfied if, for example, both $a(s)$ and $b(s)$ are monotone-increasing functions, with $b > a$. Thus, the wake commences at the section $s = 0$, and occupies the whole region $0 < x < a(s)$, $0 < s < L$. Slenderness requires that $a, b \ll L$.

The nose of the wing may be at $s = 0$; however, this is not essential, since we can simply append a 'regular' wake-free slender lifting surface with $s < 0$, to the wing under consideration. Similarly, the wing may continue for $s > L$, or else may terminate abruptly there. In this context, 'abrupt' termination, either at $s = 0$ or $s = L$, means termination within a distance in the s-direction, that is small compared to that for s-wise rates of change. This means, for example, that termination in distances comparable to the span parameters a, b, is equivalent to instantaneous termination, to leading order in slenderness. Such terminations are sketched as dashed lines in Figure 1.

The wing itself is supposed to be of zero thickness, and to have equation

$$y = \eta(x, s), \tag{1.3}$$

where η is a small quantity. We retain only $O(\eta)$ quantities from now on, using a perturbation velocity potential $\phi(x, y, s) = O(\eta)$. Thus, the linearized boundary condition on the top and bottom surfaces of the wing is, with $O(\eta^2)$ error,

$$\phi_y(x, 0_\pm, s) = U\eta_s(x, s), a(s) < x < b(s). \tag{1.4}$$

The function ϕ is odd in y, and hence vanishes at $y = 0$ except in the wing and its wake.

The corresponding linearized pressure perturbation is

$$p(x, y, s) = -\rho U \phi_s(x, y, s), \tag{1.5}$$

and hence the wake boundary condition is

$$\phi_s(x, 0_\pm, s) = 0, \quad 0 < x < a(s). \tag{1.6}$$

Upon integration of (1.6), we set

$$\phi(x, 0_\pm, s) = \pm\,\Phi(x), \quad 0 < x < a(s), \tag{1.7}$$

for some function $\Phi(x)$ to be determined. The wake is thus a vortex sheet, with strength $2\Phi(x)$ at lateral co-ordinate x.

The general incompressible lifting-surface problem is to solve the three-dimensional Laplace equation for a potential $\phi(x, y, s)$ that is an odd function of y, vanishing at infinity, and satisfying the boundary conditions (1.4), (1.6) on the plane $y = 0$. For a unique solution at arbitrary aspect ratio, we must include, in addition, the Kutta condition of continuity of pressure across the trailing edge. The slenderness, or low-aspect-ratio, simplification is in principle nothing more than neglect of streamwise derivatives in the governing equation, leaving the two-dimensional Laplace equation in the (x, y) plane, with s as a mere parameter. This approximation is also valid if the fluid is compressible, even at moderate supersonic speeds. It is notable that, in the 'classical' low-aspect-ratio theory, in which there is no wake-body interference, the Kutta condition must then be abandoned, since the solution is uniquely determined section by section, starting from the nose.

With applications to swimming of fish in mind, Newman and Wu [5] have solved slender-body problems similar to those considered here. Their work is more general, in that it treats unsteady flows and allows non-zero body thickness, but less general in allowing no spanwise distribution of camber or twist. The present problem has its motivation in planing-surface theory ([6], [7], [8]) for small-draft boats, which is equivalent to lifting-surface theory at sufficiently high Froude number. The yawed planing-surface configuration is important for high-speed boats in turns, and for surfboards [9]. Yawed wings have also been proposed for supersonic airplanes [10].

2. Solution of the cross-flow problem

We now solve the problem illustrated in Figure 2, for irrotational flow in the (x, y) plane, over a section $a < x < b$ of the wing at fixed s, and its accompanying wake $0 < x < a$. At the leading edge $x = b$, there is an inverse-square-root singularity in the fluid velocity, whereas no such singularity is tolerable at the trailing edge $x = a$. The latter requirement is, in effect, a version of the Kutta condition.

We introduce the complex function

$$W(z) = (z - b)^{\frac{1}{2}}\,(z - a)^{-\frac{1}{2}} f'(z) \tag{2.1}$$

where

$$z = x + iy, \tag{2.2}$$

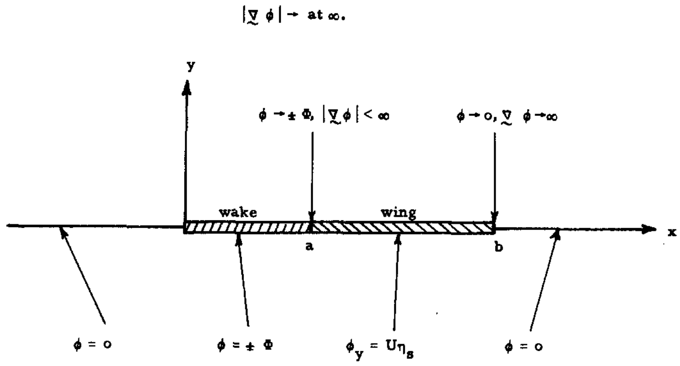

Figure 2. Cross-flow boundary-value problem.

and

$$f(x) = \phi(x, y, s) + i\psi(x, y, s) \tag{2.3}$$

is the complex velocity potential at station s. The branches of the square-root functions are such that, for example,

$$(z - b)^{\frac{1}{2}} \to \pm\, i(b - x)^{\frac{1}{2}},\ y \to 0_{\pm}. \tag{2.4}$$

Since $f'(z)$, and hence $W(z)$, tends to zero as $|z| \to \infty$, Cauchy's theorem implies that the real and imaginary parts of $W(z)$ are Hilbert transforms of each other along the x-axis. In particular, we have

$$W(x \pm i0) = \mp \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{W(\xi \pm i0)}{\xi - x} d\xi, \tag{2.5}$$

the integral being of Cauchy-principal-value form.

There is no contribution to the integral in (2.5) for $\xi < 0$ or $\xi > b$, since ϕ vanishes there, and hence so does ϕ_x and $\mathscr{R}W$. In the remainder of the x-axis,

$$W(x \pm i0) = \begin{cases} \pm \sqrt{\dfrac{b - x}{x - a}}\, (\phi_y + i\phi_x),\ x \epsilon (a, b), \\ \sqrt{\dfrac{b - x}{a - x}}\, (\phi_x - i\phi_y),\ x\epsilon(0, a), \end{cases} \tag{2.6}$$

and hence

$$\begin{aligned} \mathscr{I}W(x \pm i0) = &- \frac{1}{\pi} \int_a^b \frac{d\xi}{\xi - x} \sqrt{\frac{b - \xi}{\xi - a}}\, \phi_y(\xi, 0_{\pm}, s) \\ &\mp \frac{1}{\pi} \int_o^a \frac{d\xi}{\xi - x} \sqrt{\frac{b - \xi}{a - \xi}}\, \phi_x(\xi, 0_{\pm}, s). \end{aligned} \tag{2.7}$$

We now make use of the boundary conditions (1.5), (1.7), to give, for $x \epsilon (a, b)$,

$$\pm \phi_x(x, 0_\pm, s) = -\frac{U}{\pi}\sqrt{\frac{x-a}{b-x}}\int_a^b \frac{d\xi}{\xi - x}\sqrt{\frac{b-\xi}{\xi-a}}\,\eta_s(\xi, s) - \frac{1}{\pi}\sqrt{\frac{x-a}{b-x}}\int_o^a \frac{d\xi}{\xi-x}\sqrt{\frac{b-\xi}{a-\xi}}\,\Phi'(\xi)\,. \tag{2.8}$$

If $\Phi(\xi)$ were known, (2.8) would provide the required solution, since, from the value of the lateral velocity component ϕ_x on the wing, we can compute all forces and moments of interest. Before we can do this, however, we must determine $\Phi(\xi)$, as follows.

First we integrate (2.8) from $x = a$ to $x = b$, invoking continuity at these points, i.e. setting

$$\phi(a(s), 0_\pm, s) = \pm\, \Phi(a(s)) \tag{2.9}$$

and

$$\phi(b(s), 0_\pm, s) = 0. \tag{2.10}$$

Thus we have

$$-\Phi(a(s)) = U\int_a^b d\xi\sqrt{\frac{b-\xi}{\xi-a}}\,\eta_s(\xi, s)\, I_0(\xi) + \int_o^a d\xi\sqrt{\frac{b-\xi}{a-\xi}}\,\Phi'(\xi)\, I_0(\xi), \tag{2.11}$$

where

$$I_0(\xi) = -\frac{1}{\pi}\int_a^b \frac{dx}{\xi - x}\sqrt{\frac{x-a}{b-x}}\,. \tag{2.12}$$

This is a well-known integral ([11], p. 250) and takes the value

$$I_0(\xi) = \begin{cases} 1, & \xi \,\epsilon\, (a, b) \\ 1 - \sqrt{\dfrac{a-\xi}{b-\xi}}\,, & \xi \,\epsilon\, (0, a). \end{cases} \tag{2.13}$$

Hence (2.11) reduces to

$$-\Phi(a(s)) = U\int_a^b d\xi\sqrt{\frac{b-\xi}{\xi-a}}\,\eta_s(\xi, s) + \int_o^a d\xi\sqrt{\frac{b-\xi}{a-\xi}}\,\Phi'(\xi) - \int_o^a \Phi'(\xi)d\xi\,.$$

The quantity $-\Phi(a(s))$ cancels from both sides, leaving finally

$$\int_o^a d\xi \sqrt{\frac{b-\xi}{a-\xi}}\, \Phi'(\xi) = -U \int_a^b d\xi \sqrt{\frac{b-\xi}{\xi-a}}\, \eta_s(\xi, s). \tag{2.14}$$

Equation (2.14) is the essential result of this analysis. It is an integral equation to determine the wake lateral velocity function $\Phi'(x)$, given the wing shape $\eta(\xi, s)$. The mathematical character of this integral equation is displayed more clearly if we write

$$u(x) = \Phi'(x), \quad x = a(s),$$

and

$$b(s) = B(x), \tag{2.15}$$

giving

$$\int_o^x d\xi\, u(\xi) K(x, \xi) = g(x), \tag{2.16}$$

where

$$K(x, \xi) = \sqrt{\frac{B(x)-\xi}{x-\xi}} \tag{2.17}$$

and

$$g(x) = -U \int_x^{B(x)} d\xi \sqrt{\frac{B(x)-\xi}{\xi-x}}\, \eta_s(\xi, s). \tag{2.18}$$

When the wing geometry is prescribed by prescribing $a(s)$, $b(s)$ and $\eta(x, s)$, the functions $B(x)$, $g(x)$ and $K(x, \xi)$ are known, and (2.16) is a singular Volterra integral equation of the first kind to determine the unknown function $u(x)$. The singularity at $\xi = x$ is of an inverse-square-root nature, and hence the equation is a generalization of Abel's integral equation [12].

Except in two special cases to be discussed later, analytic solution of (2.16) is out of the question. However, a number of successful methods have been developed for numerical solution of integral equations of Abel type (e.g. [13], [14]), and we may consider that the problem is, in effect, solved, now that we have reduced it in difficulty to a computational task as straightforward as solving (2.16).

It is important to note that the Volterra character of the integral equation means that it can be solved by a marching process, starting at $x = 0$ (or $s = 0$) and moving downstream. Thus, in contrast to the case where there is no wake, sections of the wing *do* have an influence on other sections downstream. However, in opposite contrast to the lifting-surface or arbitrary-aspect-ratio case, there is no upstream influence; the flow at any section is independent of conditions downstream of it.

In the special case of untwisted or sectionally-flat wings, in which the local angle of attack

$$-\eta_s(x, s) = \alpha(s) \tag{2.19}$$

is independent of x, the integrals on the right of (2.14) or (2.18) may be evaluated explicitly. Thus (2.14) may be written

$$\int_o^a d\xi \sqrt{\frac{b-\xi}{a-\xi}}\,\Phi'(\xi) = \frac{\pi}{2}\, U\alpha(b-a), \tag{2.20}$$

which is equivalent to an integral equation derived by Newman and Wu [5], eq. (5.8)).

3. Forces and moments

If we define dF as the element of force in the y-direction on that portion of the wing between s and $s + ds$, then

$$\frac{dF}{ds} = \int_a^b [p(x, 0_-, s) - p(x, 0_+, s)]\,dx \tag{3.1}$$

$$= 2\rho U \int_a^b \phi_s(x, 0_+, s)\,dx \tag{3.2}$$

Since

$$\frac{d}{ds}\int_a^b \phi dx = \int_a^b \phi_s dx + b'\phi(b, 0_+, s) - a'\phi(a, 0_+, s)$$

$$= \int_a^b \phi_s dx - a'\Phi(a), \tag{3.3}$$

we have

$$\frac{dF}{ds} = 2\rho U \left[\frac{d}{ds}\int_a^b \phi dx + a'\Phi(a)\right]$$

i.e. $dF = F'(s)\,ds$, (3.4)

where

$$F(s) = 2\rho U \left[\int_a^b \phi(x, 0_+, s)\,dx + \int_o^a \Phi(x)\,dx\right]$$

$$= 2\rho U \int_o^b \phi(x, 0_+, s)\,dx, \tag{3.5}$$

since $\Phi(x) = \phi(x, 0_+, s)$ for $x \in (0, a)$. Equation (3.5) may be considered as a form of the Kutta-Joukowski theorem, since the complete wing-wake combination is modelled by a vortex sheet of strength $2\phi(x, 0_+, s)$ per unit lateral distance.

By an integration by parts, we have

$$F(s) = -2\rho U \int_o^b x\,\phi_x(x, 0_+, s)\,dx \tag{3.7}$$

$$\begin{aligned} &= -2\rho U^2 \int_a^b d\xi \sqrt{\frac{b-\xi}{\xi-a}}\,\eta_s(\xi, s)\,I_1(\xi) \\ &\quad -2\rho U \int_o^a d\xi \sqrt{\frac{b-\xi}{a-\xi}}\,\Phi'(\xi)\,I_1(\xi) \\ &\quad -2\rho U \int_o^a \xi\,\Phi'(\xi)\,d\xi, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} I_1(\xi) &= -\frac{1}{\pi}\int_a^b \frac{dx\; x}{\xi - x}\sqrt{\frac{x-a}{b-x}} \\ &= \frac{b-a}{2} + \xi\,I_0(\xi). \end{aligned} \tag{3.9}$$

Thus

$$\begin{aligned} F(s) = &-\rho U(b-a)\left[\int_o^a d\xi \sqrt{\frac{b-\xi}{a-\xi}}\,\Phi'(\xi) + U\int_a^b d\xi \sqrt{\frac{b-\xi}{\xi-a}}\,\eta_s(\xi, s)\right] \\ &-2\rho U\left[\int_o^a d\xi \sqrt{\frac{b-\xi}{a-\xi}}\,\xi\,\Phi'(\xi) + U\int_a^b d\xi \sqrt{\frac{b-\xi}{\xi-a}}\,\xi\eta_s(\xi, s)\right]. \end{aligned} \tag{3.10}$$

The first part of (3.10) is zero because of the integral equation (2.14), and we have

$$F(s) = F_B(s) + F_W(s) \tag{3.11}$$

where

$$F_B(s) = -2\rho U^2 \int_a^b d\xi\,\sqrt{(\xi-a)(b-\xi)}\;\;\eta_s(\xi, s), \tag{3.12}$$

and

$$F_W(s) = 2\rho U \int_o^a d\xi\,\sqrt{(a-\xi)(b-\xi)}\;\;\Phi'(\xi). \tag{3.13}$$

Upon examination of the form of the expressions (3.12), (3.13) it is seen that F_B is the lift force one obtains for a slender wing in the absence of a wake. The integral (3.12) can be evaluated uniquely and separately for each cross-section s, and is independent of the geometry at any other cross section, given only the streamwise slope or local angle of attack-η_s at the section s. On the other hand, the quantity F_W is a correction to F_B, depending on the presence of the wake, in which there is an effect at station s, indirectly through the integral equation (2.14) determining $\Phi'(\xi)$, of stations forward of s.

The net lift force on any section of the wing extending from $s = 0$ to $s = L$, in which the port side is trailing and the starboard side leading, as assumed throughout this paper, is given by $F(L) - F(0)$. The sway force in the x direction and drag force in the s direction, as well as the yaw moment about the y axis, are second-order quantities with respect to η, and will not be considered here. The remaining two moments are of first order in η, and determine the important longitudinal and lateral position of the center of pressure of the lifting forces.

The longitudinal center of pressure is determined by computation of the (nose-down) pitch moment M_P about the x-axis,

$$M_P = \int_o^L s\ dF = LF(L) - \int_o^L F(s)ds. \tag{3.14}$$

This quantity may therefore be evaluated directly, using the already-computed longitudinal lift distribution function $F(s)$.

In order to evaluate the lateral position of the center of pressure, we must return to the complete pressure distribution on the wing, to compute the (starboard-up) roll moment M_R,

$$M_P(s) = 2\rho U \int_o^b x\phi(x, 0_+, s)dx \tag{3.15}$$

$$= -\rho U \int_o^b x^2 \phi_x(x, 0_+, s)dx$$

$$= M_{RB}(s) + M_{RW}(s), \tag{3.16}$$

where

$$M_{RB}(s) = -\rho U^2 \int_a^b d\xi\ \sqrt{(\xi - a)(b - \xi)}\ \eta_s(\xi, s)\left[\xi + \frac{a+b}{2}\right] \tag{3.17}$$

and

$$M_{RW}(s) = \rho U \int_o^a d\xi\ \sqrt{(a - \xi)(b - \xi)}\ \ \Phi'(\xi)\left[\xi + \frac{a+b}{2}\right].$$

Again, M_{RB} is the roll moment for a wing without a wake, and depends only on the furthest-aft station s under consideration. For example, if the wing is symmetrical about the center point $\xi = (a + b)/2$, we have

$$M_{RB}(s) = \left(\frac{a+b}{2}\right) F_B(s), \tag{3.19}$$

i.e. the centre of pressure appears to be located at the centerline of this station. This is no longer the case when there is a wake, as evidenced by the correction introduced by the term $M_{R\,W}(s)$.

Note that, even in the absence of the wake (e.g. if the port edge is a leading edge), the centre of pressure for a wing extending from $s = 0$ to $s = L$ does not lie on the wing centre-line, since the centre of pressure is necessarily forward of the trailing edge. This means that a slender pointed wing yawed even slightly to port, tends to roll in such a way that its starboard edge rises. This asymmetry with respect to the wing center line is apparent in the section-wise pressure distribution, the strength of the port leading edge singularity being less than that of the starboard edge, and ultimately vanishing as the port edge becomes tangent to the free-stream direction.

4. Special cases

The special case of sectionally-flat wings has already been mentioned. This specialization does not lead to any essential simplification in the task of solving the integral equation (2.16). However, the resulting explicit right-hand-side in (2.20), i.e.

$$g(x) = \frac{\pi}{2}\, U\alpha(s)(B(x) - x), \qquad x = a(s), \tag{4.1}$$

may be re-interpreted in the general case (once $g(x)$ has been computed by (2.20)), as defining an 'effective' local angle of attack $\alpha(s)$, whenever (2.19) is not satisfied. The further specialization to an uncambered (i.e., *everywhere*-flat) wing, with $\alpha =$ constant, also does little to ease our task, except that now the whole problem is specified by the *single* input function $B(x)$. For definiteness, we assume henceforth that this is the case, i.e. seek to solve

$$\int_o^x d\xi\, u(\xi) \sqrt{\frac{B(x) - \xi}{x - \xi}} = \frac{\pi}{2}\, U\alpha(B(x) - x), \tag{4.2}$$

for various choices of $B(x)$, with α (and U) constant.

There appear to be only two* choices for $B(x)$ that lead to an explicit solution, namely

$$B(x) = b_o = \text{constant}, \tag{4.3}$$

and

$$B(x)/x = \coth 2\gamma = \text{constant}. \tag{4.4}$$

* However, there is a third choice, $B = b_o + x$, that reduces the problem to that of evaluating an inverse Laplace transform

The first case (4.3) corresponds to a starboard edge $x = b(s) = b_o$ = constant that is parallel to the free stream, and hence borderline between leading and trailing. Thus, the classical [1] slender wing theory should apply, and, for everywhere-flat wings, indicate zero loading on these sections of the wing.

The corresponding simplification to the integral equation reduces it to an Abel equation, with explicit solution

$$u(x) = \frac{1}{2}\, U\alpha \frac{b_o - 2x}{\sqrt{x(b_o - x)}}, \tag{4.5}$$

or

$$\Phi(x) = U\alpha \sqrt{x(b_o - x)}. \tag{4.6}$$

In fact, in this case $\phi(x, 0_{\pm}, s) = \pm\, \Phi(x)$ for all x, $0 < x < b_o$, so that even on the wing $a < x < b_o$ itself we have $\phi_s = 0$, and the loading vanishes. Alternatively, we find that (3.12) and (3.13) give equal and opposite lift contributions, so that the total in zero. If α is not constant, the solution of the Abel equation (4.2) subject to (4.3) can be written down in terms of a single quadrature.

The special case (4.4) is somewhat less trivial. This case includes, as a further specialization, a simple yawed delta wing of triangular planform, with

$$a(s) = \theta_1 s, \tag{4.7}$$

and

$$b(s) = \theta_2 s, \tag{4.8}$$

i.e.

$$B(x) = \frac{\theta_2 x}{\theta_1}. \tag{4.9}$$

Thus the delta wing has a nose angle $\theta_2 - \theta_1$, and is yawed at an angle $(\theta_1 + \theta_2)/2$ to the stream. The ratio between leading and trailing edge angles defines the constant γ in (4.4), i.e.

$$\frac{\theta_2}{\theta_1} = \coth 2\gamma. \tag{4.10}$$

More generally, (4.4) is true whenever the leading and trailing edges have the same shape, whether or not that shape is a straight line; thus it applies also to some 'banana' or 'scimitar' configurations.

If (4.4) holds, it is clear that the wake strength is constant, i.e. (4.2) possesses a solution with $u(x)$ = constant, specifically

$$u(x) = \frac{\frac{\pi}{2}\, U\alpha}{\gamma + \sinh\gamma \cosh\gamma}. \tag{4.11}$$

The resulting forces are

$$F_B = \frac{1}{4}\pi\rho U^2\alpha(b-a)^2, \tag{4.12}$$

and

$$F_W = \mu F_B, \tag{4.13}$$

where

$$\mu = (\tfrac{1}{4}\sinh 4\gamma - \gamma)/(\gamma + \sinh\gamma\cosh\gamma). \tag{4.14}$$

Note that since $1 < 1+\mu < \cosh 4\gamma$, the lift lies between that obtained by ignoring the wake, and that obtained by 'filling it in' with a solid surface. This is illustrated in Figure 3, where we plot $1+\mu$ against $\cosh 2\gamma$. The abscissa

$$\cosh 2\gamma = (\theta_1 + \theta_2)/(\theta_2 - \theta_1) \tag{4.15}$$

measures yaw angle/apex half angle, for a triangular delta wing.

The corresponding roll moments are

$$M_{RB} = \frac{a+b}{2} F_B, \tag{4.16}$$

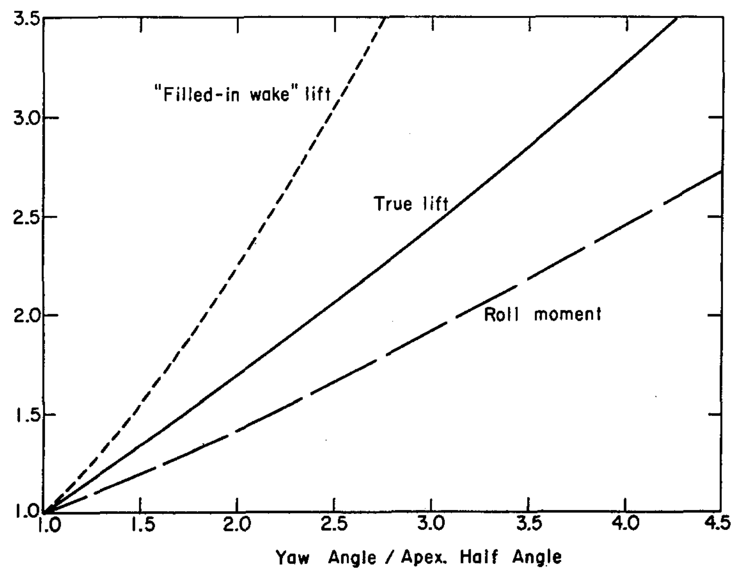

Figure 3. Lift force and roll moment for triangular wings as functions of yaw angle. ——— : Lift/Lift at zero yaw. – – – – : Lift with filled-in wake/Lift at zero yaw. — — — : Roll moment/Roll moment at zero yaw.

and

$$M_{RW} = \nu M_{RB}, \tag{4.17}$$

where

$$\nu = (\tfrac{1}{6} \sinh 4\gamma + \tfrac{1}{6} \tanh 2\gamma - \gamma)/(\gamma + \sinh\gamma \cosh\gamma). \tag{4.18}$$

In (4.16) (and (4.12)), a and b are evaluated at the aft-most station $s = L$ of the wing. Figure 3 also shows $1 + \nu$ plotted against $\cosh 2\gamma$.

The total pitch moment M_P is scaled by the same factor $1 + \mu$ as the total lift force F, in this case. Hence, the presence of the wake does not change the *longitudinal* location of the centre of pressure. For example, the centre of pressure on a triangular delta wing remains at the 2/3 chord point, for all yaw angles.

On the other hand, since $\nu \neq \mu$, the presence of the wake does shift the centre of pressure laterally, from $x = (a + b)/2$ to

$$x = \frac{a+b}{2}\left[\frac{1+\nu}{1+\mu}\right] \tag{4.19}$$

This shift is toward the wake, since $\mu > \nu$, and thus tends to reduce the roll moment. In the limit as $\gamma \to \infty$, the ratio in (4.19) tends to 3/4, which means that, at least for triangular delta wings, the centre of pressure moves to the wing centre line, at the 2/3 chord point. However, for all finite yaw angles, there remains a starboard-up roll moment about the wing centre line, due to yaw to port, whether or not there is a wake.

The ultimately-linear increase of μ and ν with $\cosh 2\gamma$ as $\gamma \to \infty$ in Figure 3, indicates an approach to a lifting-line type of theory as $(b - a)/a \to o$, in which the net forces and moments can be computed in a strip-wise manner, using two-dimensional results at fixed x. For example, in the present case as $\gamma \to \infty$, we find $F_B \to 0$ and

$$F \to F_W \to \frac{1}{2}\pi\rho U^2 \alpha a(b - a), \tag{4.20}$$

which varies linearly with the yaw angle. It is possible that in this limit there is an equivalence of the present theory to the oblique lifting-line theory of Cheng [15], although this has not yet been demonstrated.

5. Numerical solution for straight-edged wings

For general input functions $B(x)$, a numerical scheme for solution of integral equations of the form (4.2) has been described by Tuck [14]. In essence, this procedure simply replaces integration by summation over a uniformly-spaced mesh, using a special set of integration weights that account for the inverse-square-root singularity at $\xi = x_-$. In cases where the expected solution

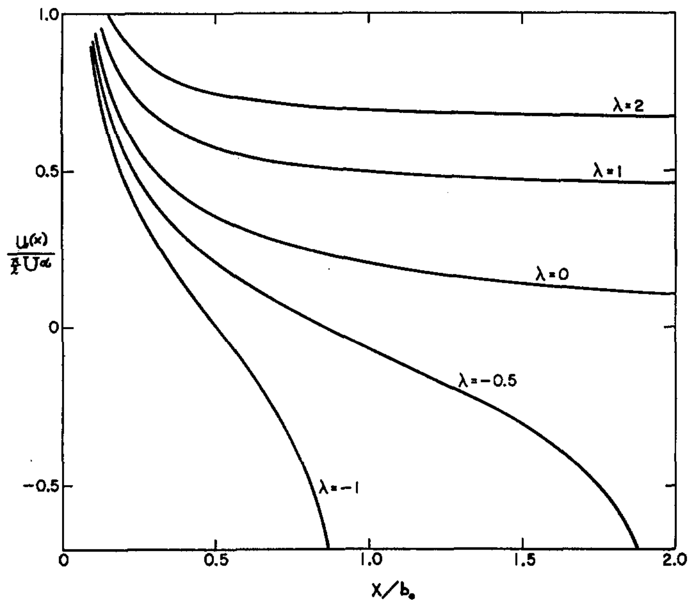

Figure 4. Wake strength (lateral velocity jump) development for a family of straight-edged wings, parametrized by λ.

$u(\xi)$ is singular at $\xi = 0_+$ (c.f. (4.5)), it is also possible to make allowance for this singularity. The resulting discrete system of linear algebraic equations has a triangular matrix, and the solution is immediate. Four-to-five figure accuracy is achieved, using an x-wise spacing of 0.025 $a(L)$.

A general family of straight-edged wings can be described by

$$B(x) = b_o + (\lambda + 1)x, \tag{5.1}$$

for constants b_o, λ. The condition that $x = b(s)$ be a trailing edge demands that $\lambda \geqslant -1$. The special case (4.3) corresponds to $\lambda = -1$, while the special case (4.4) is $b_o = 0$, $\lambda = \operatorname{cosech} 2\gamma$. If $-1 \leqslant \lambda < 0$, the wing terminates at $x = -b_o/\lambda$, at a sharp tail point where $a(s) = b(s)$. If $\lambda = 0$, the wing is of constant span, whereas if $\lambda > 0$, the span increases indefinitely downstream. In the case $\lambda > 0$, if we go sufficiently far downstream, i.e. let x (or s) $\to \infty$, the results become insensitive to the initial span b_o, and we recover the delta-wing reults of the previous section, with $\lambda = \operatorname{cosech} 2\gamma$.

Results computed for $u(x)$ on a TRS-80 microcomputer are shown in Figure 4, and confirm the above features. An inverse-square-root singularity is always present at $x = 0$, i.e. along the track of the nose. The wing with $b_o \neq 0$ commences instantaneously at $x = 0$ with non-zero span. However, as mentioned in Sec. 1, in the (over-all) low-aspect ratio limit, the same results also hold, to leading order, if there is a short (chord $= O(b_o)$) finite-aspect-ratio pointed airfoil appended for $x < 0$. In the finite chord case $\lambda < 0$, there is also a (negative) inverse-square-root singularity at the tail point. On the other hand, for $\lambda \geqslant 0$, the wake velocity tends rapidly to the constant delta-wing value (4.11) as $x \to \infty$.

Figure 5 shows the lift force F, and Figure 6 the centre of pressure $(\bar{s}, \bar{x})$ in the special case

$\lambda = 0$ of a rectangular wing of constant width b_o. The abscissa is the same quantity x/b used in Figure 4, but now with $x = a(L)$, and thus measures the angle of yaw $a(L)/L$, scaled with respect to the wing's aspect ratio b_o/L. Again, the lift lies between the no-wake (zero-yaw) value and the 'filled-in-wake' value, shown dashed, but is rather closer to the latter value than for the delta-wing of Figure 3. As the yaw angle increases, the lift becomes asymptotically a linear function of yaw angle, as would be predicted by a strip-wise formula similar to (4.20).

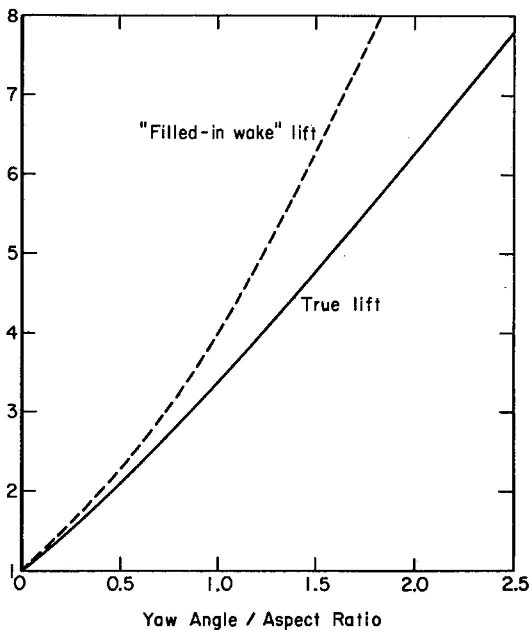

Figure 5. Lift force on a rectangular wing as a function of yaw angle.

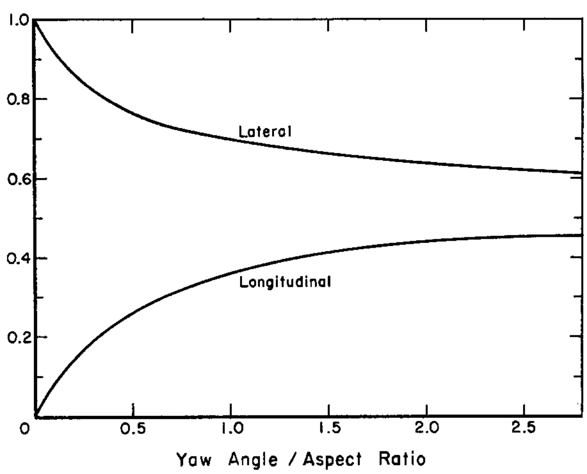

Figure 6. Centre of pressure $(\bar{s}, \bar{x})$ of a rectangular wing as a function of yaw angle. Longitudinal position $\bar{s}$ scaled with respect to length L; lateral position $\bar{x}$ with respect to mid-point $a(L) + \frac{1}{2} b_o$ of trailing edge.

The centre of pressure moves from the mid-point of the leading edge, to the geometric centre of the wing, as the yaw angle increases from zero. This is displayed in Figure 6 by scaling its longitudinal position $s = \bar{s}$ by the length L, and lateral position $x = \bar{x}$ by the co-ordinate $x = a(L) + \frac{1}{2} b_o$ of the mid-point of the trailing edge. Both plotted quantities tend to 1/2 as the yaw angle becomes large.

Note that the special case chosen in Figure 6 is the one in which the problem can be reduced to an inverse Laplace transform. In fact, (c.f. Mirels, [2]), in this case there is an equivalence to two-dimensional unsteady flow in the x, y plane, with the s-co-ordinate playing a time-like role, and it should be possible to relate the results obtained to those for the so-called Wagner problem [16].

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